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EFFECTIVE PERMEABILITIES IN A POROUS MEDIUM WITH LOG-STABLE STATISTICS

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Equations for the effective coefficients of random permeability fields for fluid flow through a porous medium with log-stable distributions are derived using the Wilson renormalization group approach. Results of the theoretical modeling are compared with data of numerical modeling.

Key words: *filtration, log-stable permeability distribution, subgrid modeling.*

Introduction. Full-scale and laboratory observations have shown that the fluid permeability field is nonuniform and that the spread of the nonuniformity scale increases with an increase in the number of measurements in a bounded interval. This has led to the development of fractal permeability models with log-stable permeability distributions [1]. Shvidler [2] and Dagan [3] studied statistical models with lognormal permeability distributions. Kuz'min and Soboleva [4, 5], using the Wilson renormalization group (RG) approach [6], derived subgrid formulas for the effective permeability coefficients and studied the diffusion of the interface between the fluids for their joint flow in a multiscale porous medium for lognormal permeability and porosity distributions. Teodorovich [7] derived the Landau–Lifshits formula for the effective permeability within the framework of a rigorous field renormalization group and analyzed previous studies of the field RG. In particular, mention is made of the arguments of [8], according to which renormalization group methods partially allow for the highest orders of perturbation theory and should improve the accuracy of the formulas derived. The same arguments are also applicable to subgrid modeling. If the medium is assumed to satisfy the improved Kolomogorov similarity hypothesis [9, 4], the effective coefficients take especially simple form. In the present paper, the ideas of the Wilson RG method are employed to find subgrid modeling formulas for solving filtration problems for fractal porous media with a lognormal permeability distribution. Differential equations for obtaining effective constants are also derived for media that do not satisfy the improved similarity hypothesis. The derived formulas are verified by direct numerical modeling.

Formulation of the Problem. At small Reynolds numbers, the filtration velocity $\boldsymbol{v}$ and the pressure p are related by the Darcy law $\boldsymbol{v} = -\varepsilon(\boldsymbol{x})\nabla p$, where $\varepsilon(\boldsymbol{x})$ is the permeability (a random function of the coordinates). For an incompressible fluid, we have the equation

$$\nabla[\varepsilon(\boldsymbol{x})\nabla p(\boldsymbol{x})] = 0. \tag{1}$$

Let the permeability field be known. This implies that permeability measurements are performed at each point $\boldsymbol{x}$ by pressing the fluid through a sample of very small size l_0. The random function of spatial coordinates $\varepsilon(\boldsymbol{x})$ is treated as the permeability limit: $\varepsilon(\boldsymbol{x})_{l_0} \to \varepsilon(\boldsymbol{x})$ at $l_0 \to 0$. To pass to a coarser grid l_1, one cannot simply smooth the obtained field $\varepsilon(\boldsymbol{x})_{l_0}$ on the scale $l_1 > l_0$. The resulting field is not the true permeability which describes filtration in the scale field (l_1, L), where L is the largest scale. To find the permeability on a coarser grid, it is necessary to perform new measurements, pressing the fluid through larger samples of size l_1. The need for this procedure is due to the fact that permeability fluctuations from the range of (l_0, l_1) correlate with their induced pressure fluctuations. Similarly to [9], we consider the nondimensional field ψ equal to the ratio of the

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permeabilities smoothed on two different scales (l, l_1). A detailed description of this approach is given in [4]. Let $\tilde{\varepsilon}(\boldsymbol{x})_l$ be the permeability $\varepsilon(\boldsymbol{x})_{l_0}$ smoothed on the scale l; then, $\psi(\boldsymbol{x}, l, l_1) = \tilde{\varepsilon}(\boldsymbol{x})_{l_1}/\tilde{\varepsilon}(\boldsymbol{x})_l$. As $l_1 \to l$, we obtain the field $\varphi(\boldsymbol{x},l) = d\psi(\boldsymbol{x}, l, l\lambda)/(l\, d\lambda)\Big|_{\lambda=1}$ which defines all statistical properties of the porous medium. The obtained relation is a differential equation, whose solution gives the permeability as a function of the field φ:

$$\varepsilon(\boldsymbol{x})_{l_0} = \varepsilon_0 \exp\Big[-\int\limits_{l_0}^{L} \varphi(\boldsymbol{x}, l)\, \frac{dl}{l}\Big]. \tag{2}$$

It is assumed that the permeability is characterized by nonuniformity of the scale l from the interval $l_0 < l < L$ and the field φ is isotropic and homogeneous. For any $\boldsymbol{x}$ and $\boldsymbol{y}$, the fields $\varphi(\boldsymbol{x}, l)$ and $\varphi(\boldsymbol{y}, l')$ with different scales are statistically independent. This hypothesis is usually assumed to be valid in different models and reflects the fact that statistical dependence decays when the scale fluctuation become different in magnitude [9]. The scale properties of the model are defined by the field $\varphi(\boldsymbol{x}, l)$. For scale-invariant models, the condition $\varphi(\boldsymbol{x}, l) \to \varphi(K\boldsymbol{x}, Kl)$ should be satisfied. According to the theorem of the sums of independent random fields [10], if the variance $\varphi(\boldsymbol{x}, l)$ at a given point is finite, then for large values of L/l_0, the integral in (2) tends to a normal field; when the variance is infinite and there exists a nondegenerate (not concentrated at one point) limiting distribution of the sum of random variables, this distribution is necessarily stable. For simplification, it is assumed that the field $\varphi(\boldsymbol{x}, l)$ has a stable distribution.

Log-Stable Model. An increase in the randomness and intermittency in the behavior of the physical fields with an increase in the measurement scale has forced many researchers to reject the lognormal model and consider the more general case of log-stable distributions [11, 12]. Bouffadel et al. [1], using experimental data for boreholes, obtained distributions of permeability fields and some statistical characteristics and showed that the permeability fields can obey log-stable distributions.

The stable distributions depend on four parameters: α, β, μ, and σ [13]. The parameter α is in the range of $0 < \alpha \le 2$, where $\alpha = 2$ corresponds to a Gaussian distribution. For $m \ge \alpha$, the statistical moments of order m do not exist except for the case $\alpha = 2$, for which all statistical moments are determined. Thus, the variance is infinite for $\alpha < 2$ and the average value of a random variable is infinite for $\alpha < 1$. The parameters β, μ, and σ (no matter to which form of representation of the stable distribution they are related) characterize the asymmetry, shift, and scale of the distribution, respectively. The parameter β varies in the range of $[-1, 1]$. For $\beta = 0$, the distribution is symmetric, the value $\beta = -1$ corresponds to an asymmetric distribution in which the probability of occurrence of large negative values of random variables is high; the other extreme case is $\beta = 1$. The parameter σ is equal to half the variance for $\alpha = 2$ and can serve as a measure of the distribution width for $\alpha < 2$. Figure 1 gives the one-dimensional probability densities of the stable laws for $\mu = 0$. The field $\varphi(\boldsymbol{x}, l)$ with a stable distribution law is modeled using the approach of [14, p. 69, formula 2.3.6]. At the points $(\boldsymbol{x}_j, l)$, the field φ is expressed in terms of the sum of random independent variables which have stable distributions with identical parameters α, β, and $\mu = 0$ and $\sigma = 1$ (form A is considered [13]):

$$\varphi(\boldsymbol{x}_j, l) = \Big(\frac{\Phi_0(l)}{2(\delta\tau \ln 2)^{\alpha-1}}\Big)^{1/\alpha} a^l_{\boldsymbol{ji}} \zeta^l_{\boldsymbol{i}} + \varphi_0(l). \tag{3}$$

Here $l = 2^\tau$ and $\delta\tau$ is the discretization interval over the logarithm of the scale; the coefficients $a^l_{\boldsymbol{ji}}$ have a support of size l^3, depend only on the modulus of the difference of the indices $a^l_{\boldsymbol{ji}} \equiv a^l(|\boldsymbol{i} - \boldsymbol{j}|)$; therefore, the subscript j can be omitted below. For all l, the condition $\sum\limits_{k_x}\sum\limits_{k_y}\sum\limits_{k_z}(a^l_{k_x k_y k_z})^\alpha = 1$ is satisfied. For $1 \le \alpha \le 2$, the thus constructed field φ is stable, homogeneous, and isotropic in the spatial variables [14]. If the coefficients $a^l_{\boldsymbol{ji}}$ satisfy the condition $a^l_{\boldsymbol{ji}} \equiv a^l(|\boldsymbol{i} - \boldsymbol{j}|/l)$ and if the constants $\Phi_0(l)$ and $\varphi_0(l)$ are identical for all l, the field φ is invariant under scale transformations. The average value of the field φ exists and is equal to $\varphi_0(l)$; as regards the second moments, for $\alpha \ne 2$, they are infinite. This hinders the implementation of a correlation analysis, which is performed, for example, in [4], and the use of the approach described in [7]. Nevertheless, for the extreme point $\beta = 1$, the second moments for the permeability field exist despite the absence of variance of the field φ. This case is of interest because it was confirmed by experiments [1]. Since l_0 is the minimum scale, we can set $\varepsilon(\boldsymbol{x}) = \varepsilon(\boldsymbol{x})_{l_0}$. Thus, in the model described above, the permeability field $\varepsilon(\boldsymbol{x})$ has the form

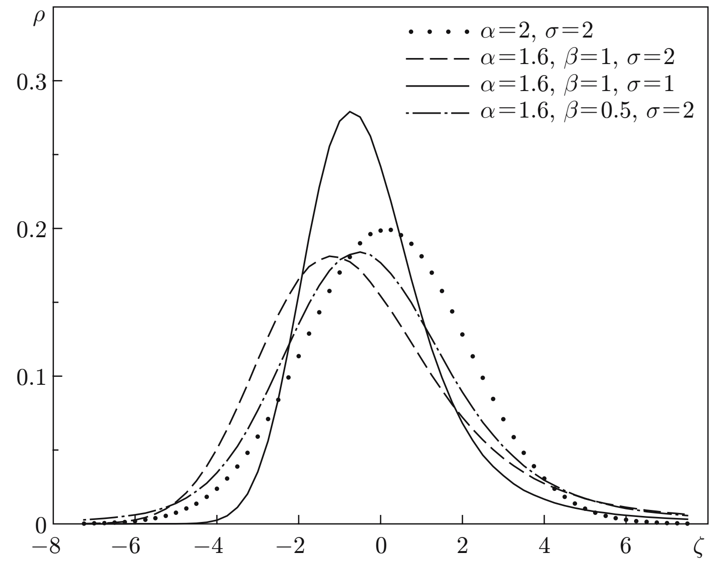

Fig. 1. One-dimensional functions of the probability densities of stable laws for $\mu = 0$ (ζ are the random variable values and $\alpha = 2$ corresponds to a normal distribution).

$$\varepsilon(\boldsymbol{x}) = \varepsilon_0 \exp\Big[-\Big(\ln 2 \sum_{\hat{l}_0}^{\hat{L}} \varphi(\boldsymbol{x}, \tau_l)\delta\tau\Big)\Big], \tag{4}$$

where $L = 2^{\hat{L}\delta\tau}$, $l_0 = 2^{\hat{l}_0\delta\tau}$, and the integral in formula (2) is replaced by a sum. The moments of the first and second orders a calculated in the Appendix. Let us consider the permeability correlation function

$$\langle \varepsilon(\boldsymbol{x})\varepsilon(\boldsymbol{x}+\boldsymbol{r})\rangle = \varepsilon_0^2 \Big\langle \exp\Big[-\sqrt[\alpha]{\frac{1}{2}\,\delta\tau \ln 2} \sum_{\hat{l}_0}^{\hat{L}} \sqrt[\alpha]{\Phi_0(\hat{l})} \sum_k (a^l_{\boldsymbol{k}}\xi^l_k + a^l_{\boldsymbol{k}}\xi^l_{\boldsymbol{k}+\boldsymbol{k}_r}) - 2\varphi_0(\hat{l})\Big]\Big\rangle. \tag{5}$$

The sum in the exponent in (5) is divided with respect to the scale $\hat{l}$ into groups $\hat{l}_0 \le \hat{l} \ll \hat{l}_r$ and $\hat{l}_r < \hat{l} < \hat{L}$, where $r = 2^{\hat{l}_r\delta\tau}$. In the estimation, the first group gives the constant C. For the second group, formula (A2) from the Appendix is used. In this case, it is taken into account that for $\zeta_{\boldsymbol{k}}$, the parameters in form A are as follows: $\mu = 0$, $\sigma = 1$, $\beta = 1$, and $1 < \alpha \le 2$. The average of the permeability is calculated by formula (A3). Finally, we obtain the estimate

$$\langle \varepsilon(\boldsymbol{x})\varepsilon(\boldsymbol{x}+\boldsymbol{r})\rangle \simeq C \exp\Big[2\delta\tau \ln 2\Big(-2^{\alpha-2} \sum_{\hat{l}_r}^{\hat{L}} \Phi_0(\hat{l})\Big[\cos\Big(\frac{\pi}{2}\,\alpha\Big)\Big]^{-1} - \varphi_0(\hat{l})\Big)\Big].$$

For a scale-invariant medium, the second group makes the power-law contribution

$$\langle \varepsilon(\boldsymbol{x})\varepsilon(\boldsymbol{x}+\boldsymbol{r})\rangle \simeq C \exp[2(-2^{\alpha-2}\Phi_0[\cos(\pi\alpha/2)]^{-1} - \varphi_0)(\ln L - \ln r)]$$

$$\simeq C(L/r)^{-2(2^{\alpha-2}\Phi_0[\cos(\pi\alpha/2)]^{-1} + \varphi_0)}. \tag{6}$$

The constant C is not universal, and the exponent in (6) for a fractal medium is universal and, according to [15], can be measured.

Subgrid Model. For a sufficiently wide spread of the scales ($L/l_0 \gg 1$), pressure estimation using Eq. (1) is impossible or requires a large volume of calculations. Therefore, it is required to obtain the effective coefficients in the equations for the large-scale filtration components. The ideas of the Wilson RG method are used to find subgrid modeling formulas. This approach is described at length in [4] for a lognormal permeability model. The permeability function $\varepsilon(\boldsymbol{x}) = \varepsilon(\boldsymbol{x})_{l_0}$ is separated into two components with respect to the scale l. The large-scale component $\varepsilon(\boldsymbol{x}, l)$ is obtained by statistical averaging over all $\varphi(\boldsymbol{x}, l_1)$ with $l_1 < l$, and the small-scale component is equal to $\varepsilon'(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}) - \varepsilon(\boldsymbol{x}, l)$:

$$\varepsilon(\boldsymbol{x}, l) = \varepsilon_0 \exp\Big[-\int\limits_{l}^{L} \varphi(\boldsymbol{x}, l_1)\,\frac{dl_1}{l_1}\Big]\Big\langle \exp\Big[-\int\limits_{l_0}^{l} \varphi(\boldsymbol{x}, l_1)\,\frac{dl_1}{l_1}\Big]\Big\rangle_<, \tag{7}$$

$$\varepsilon'(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}, l)\Big[\exp\Big[-\int\limits_{l_0}^{l} \varphi(\boldsymbol{x}, l_1)\,\frac{dl_1}{l_1}\Big]\Big/\Big\langle \exp\Big[-\int\limits_{l_0}^{l} \varphi(\boldsymbol{x}, l_1)\,\frac{dl_1}{l_1}\Big]\Big\rangle_< - 1\Big], \tag{8}$$

where $\langle\,\rangle_<$ denotes averaging over $\varphi(\boldsymbol{x}, l_1)$ for the small scale l_1. The large-scale (supergrid) pressure component $p(\boldsymbol{x}, l)$ is obtained as the averaged solution of Eq. (1) in which the large-scale component $\varepsilon(\boldsymbol{x}, l)$ is fixed and the small-scale component ε' is a random variable; $p(\boldsymbol{x}, l) = \langle p(\boldsymbol{x})\rangle_<$. The subgrid component is $p' = p(\boldsymbol{x}) - p(\boldsymbol{x}, l)$. Substituting the expression for $p(\boldsymbol{x})$ and $\varepsilon(\boldsymbol{x})$ into Eq. (1) and averaging over the small-scale component, we obtain

$$\nabla[\varepsilon(\boldsymbol{x}, l)\nabla p(\boldsymbol{x}, l) + \langle \varepsilon'(\boldsymbol{x})\nabla p'(\boldsymbol{x})\rangle] = 0. \tag{9}$$

The second term in Eq. (9) is unknown. It can not be discarded without tentative estimation since the correlation between the permeability and the pressure gradient can be substantial [2]. The choice of the form of the second term in (9) determines the subgrid model. This expression is estimated using perturbation theory. In the Wilson RG method, the initial value of the scale l is close to the smallest scale l_0, which makes it possible to obtain a differential equation for the scale l. Subtracting (9) from (1), we obtain the subgrid equation for the pressure $p'(\boldsymbol{x})$:

$$\nabla[\varepsilon(\boldsymbol{x})\nabla p(\boldsymbol{x})] - \nabla[\varepsilon(\boldsymbol{x}, l)\nabla p(\boldsymbol{x}, l) + \langle \varepsilon'(\boldsymbol{x})\nabla p'(\boldsymbol{x})\rangle] = 0. \tag{10}$$

This equation is used to find the subgrid pressure $p'(\boldsymbol{x})$. It cannot be solved exactly. From the subgrid equation (10), ignoring terms of the second order of smallness, we obtain

$$\Delta p'(\boldsymbol{x}) = -\frac{1}{\varepsilon(\boldsymbol{x}, l)}\,\nabla\varepsilon'(\boldsymbol{x})\nabla p(\boldsymbol{x}, l). \tag{11}$$

The variables $\varepsilon(\boldsymbol{x}, l)$ and $p(\boldsymbol{x}, l)$ on the right of Eq. (11) are considered known, according to the Wilson RG method, and their derivatives vary more slowly than $\varepsilon'(\boldsymbol{x})$. This meets the aim to derive a proper supergrid equation in the large-scale limit (in the Fourier representation, in the limit of small wavenumbers). Therefore,

$$p'(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon(\boldsymbol{x}, l)}\int \frac{1}{r}\,\nabla\varepsilon'(\boldsymbol{x}')\,d\boldsymbol{x}'\,\nabla p(\boldsymbol{x}, l), \tag{12}$$

where $r = |\boldsymbol{x} - \boldsymbol{x}'|$. For the pressure gradient from (12) in the large-scale limit,

$$\nabla p'(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon(\boldsymbol{x}, l)}\,\nabla\int \frac{1}{r}\,\nabla\varepsilon'(\boldsymbol{x}')\,d\boldsymbol{x}'\,\nabla p(\boldsymbol{x}, l). \tag{13}$$

Using the obtained equalities, we arrive at the following expression for the subgrid term in Eq. (9):

$$\langle \varepsilon'(\boldsymbol{x})\nabla p'(\boldsymbol{x})\rangle \approx \frac{1}{4\pi D\varepsilon(\boldsymbol{x}, l)}\int \Delta\,\frac{1}{r}\,\langle \varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x}')\rangle\,d\boldsymbol{x}'\,\nabla p(\boldsymbol{x}, l)$$

$$\approx -\frac{1}{D\varepsilon(\boldsymbol{x}, l)}\,\langle \varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x})\rangle\nabla p(\boldsymbol{x}, l),$$

where D is the dimension of the space; in this case $D = 3$. The model obtained in the present paper is similar to that proposed in [16] to calculate the effective dielectric constant of a mixture under simplifying assumptions on the smallness of fluctuations and their spatial scale.

According to formulas (7) and (A3),

$$\varepsilon(\boldsymbol{x}, l) = \varepsilon_0 \exp\Big[-\delta\tau\ln 2\Big(\sum_{\hat{l}_1=\hat{l}}^{\hat{L}} \varphi(\boldsymbol{x}, \tau_{\hat{l}_1}) + \sum_{\hat{l}_1=\hat{l}_0}^{\hat{l}} \frac{\Phi_0(\tau_{\hat{l}_1})}{2}\Big[\cos\Big(\frac{\pi}{2}\,\alpha\Big)\Big]^{-1} + \varphi_0(\tau_{\hat{l}_1})\Big)\Big]. \tag{14}$$

Using formulas (8) and (14), we obtain the subgrid component ε':

$$\varepsilon'(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}, l)\Big[\exp\Big(\delta\tau\ln 2\sum_{\hat{l}_1=\hat{l}_0}^{\hat{l}} -\varphi(\boldsymbol{x}, \tau_{\hat{l}_1}) + \frac{\Phi_0(\tau_{\hat{l}_1})}{2\cos(\pi\alpha/2)} + \varphi_0(\tau_{\hat{l}_1})\Big) - 1\Big].$$

Then,

$$\langle \varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x})\rangle = \varepsilon^2(\boldsymbol{x}, l)\Big[\exp\Big(\delta\tau \ln 2 \sum_{\hat{l}_1=\hat{l}_0}^{\hat{l}} -\varphi(\boldsymbol{x}, \tau_{\hat{l}_1}) + \frac{\Phi_0(\tau_{\hat{l}_1})^{-1}}{2\cos(\pi\alpha/2)} + \varphi_0(\tau_{\hat{l}_1})\Big) - 1\Big]^2.$$

From formula (14), retaining terms of only the first order of smallness with respect to $\delta\tau$, we obtain

$$\langle \varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x})\rangle \approx \varepsilon^2(\boldsymbol{x}, l)\delta\tau \ln 2\,[\cos(\pi\alpha/2)]^{-1}(1 - 2^{\alpha-1})\Phi_0(\tau_{\hat{l}}). \tag{15}$$

Using formulas (14) and (15), we arrive at the following formula for the second term in Eq. (9):

$$\langle \varepsilon'(\boldsymbol{x})\nabla p'(\boldsymbol{x})\rangle_< \approx -(\delta\tau \ln 2/D)[\cos(\pi\alpha/2)]^{-1}(1 - 2^{\alpha-1})\Phi_0(\tau_{\hat{l}})\varepsilon(\boldsymbol{x}, l)\nabla p(\boldsymbol{x}, l). \tag{16}$$

Substituting (16) into (9) and retaining terms of only the first order of smallness, we have

$$\nabla\Big[\Big(1 - \delta\tau \ln 2\Big(\Phi_0(\tau_{\hat{l}})\,\frac{2(1 - 2^{\alpha-1}) + D}{2D\cos(\pi\alpha/2)} + \varphi_0(\tau_{\hat{l}})\Big)\Big)\varepsilon(\boldsymbol{x}, l)\nabla p(\boldsymbol{x}, l)\Big] = 0.$$

In subgrid modeling, the effective permeability coefficient on the scale l should adequately describe the behavior of the solution of Eq. (1) in the interval of scales (l, L), i.e., after discarding the scales from the interval (l_0, l), and it should be calculated by a formula of the same form as the coefficient $\varepsilon(\boldsymbol{x}) = \varepsilon(\boldsymbol{x})_{l_0}$. The effective permeability is found from the formula

$$\varepsilon(\boldsymbol{x}) = \varepsilon_{0l} \exp\Big[-\Big(\ln 2 \sum_{\hat{l}}^{\hat{L}} \varphi(\boldsymbol{x}, \tau_{\hat{l}})\delta\tau\Big)\Big],$$

where

$$\varepsilon_{0l} = \varepsilon_0\Big[1 - \delta\tau \ln 2\Big(\Phi_0(\tau_{\hat{l}_1})\,\frac{2(1 - 2^{\alpha-1}) + D}{2D\cos(\pi\alpha/2)} + \varphi_0(\tau_{\hat{l}})\Big)\Big].$$

Replacing the variable τ by l and letting δl go to zero, we obtain the equation

$$\frac{d\ln\varepsilon_{0l}}{d\ln l} = -\Phi_0(l)\frac{2(1 - 2^{\alpha-1}) + D}{2D\cos(\pi\alpha/2)} - \varphi_0(l), \qquad \varepsilon_{0l}\Big|_{l=L} = \varepsilon_{00}. \tag{17}$$

For a scale-invariant medium, the solution of Eq. (17) has the form

$$\varepsilon_{0l} = \varepsilon_{00}(l/L)^{\Phi_0(2(1-2^{\alpha-1})+D)/(2D\cos(\pi\alpha/2))+\varphi_0}. \tag{18}$$

The constant ε_{00} describes the motion of the medium on the largest scale for $l = L$.

Numerical Modeling. Equation (1) is solved numerically in a cube with edge L_0. On the edges of the cube $y = 0$ and $y = L_0$, the pressure is set constant: $p(x, y, z)\big|_{y=0} = p_1$, $p(x, y, z)\big|_{y=L_0} = p_2$, $p_1 > p_2$. On the other edges of the cube, the pressure is specified by the linear relation for y: $p = p_1 + (p_2 - p_1)/L_0$. The main filtration flow is directed along the y axis. In the calculations, nondimensional variables are used. All distances are measured in the units of L_0, the difference $p_1 - p_2$ is used as the unit of the pressure difference, and the permeability is measured in the units of ε_0. Thus, the problem is solved in a unit cube with a single pressure jump and $\varepsilon_0 = 1$. The permeability field is modeled by formula (4). For the spatial variables, we use a $256 \times 256 \times 256$ grid with a scale step $\delta\tau = 1$, $\tau_l = 0, \ldots, -8$. The coefficients $a^l_{\boldsymbol{ji}}$ are chosen in the form

$$a^l_{\boldsymbol{ji}} = \Big(\frac{\sqrt{\alpha}}{2^{\tau_l}\sqrt{\pi}}\Big)^{3/\alpha} \exp\Big(-\frac{(\boldsymbol{x_j} - \boldsymbol{x_i})^2}{2^{2\tau_l}}\Big).$$

The field $\varphi(\boldsymbol{x}, \tau_l)$ is generated separately for each τ_l. The overall exponent in (4) is summed over statistically independent layers. In particular problems, the scales L and l_0 can take different values. In the present study, they are not rendered concrete since the goal is to find a universal subgrid model and its universal exponents of the type of the exponent in formula (18). For approximated calculations, it is possible to use a certain limited number of layers. In our case, they are three: $\tau_l = -4$, -5, and -6. The number of layers and scales are chosen as follows. The scale of the largest permeability fluctuations is chosen such that approximately probabilistic averages can be replaced by averages over space, and the scale of the smallest permeability fluctuations is chosen such that the

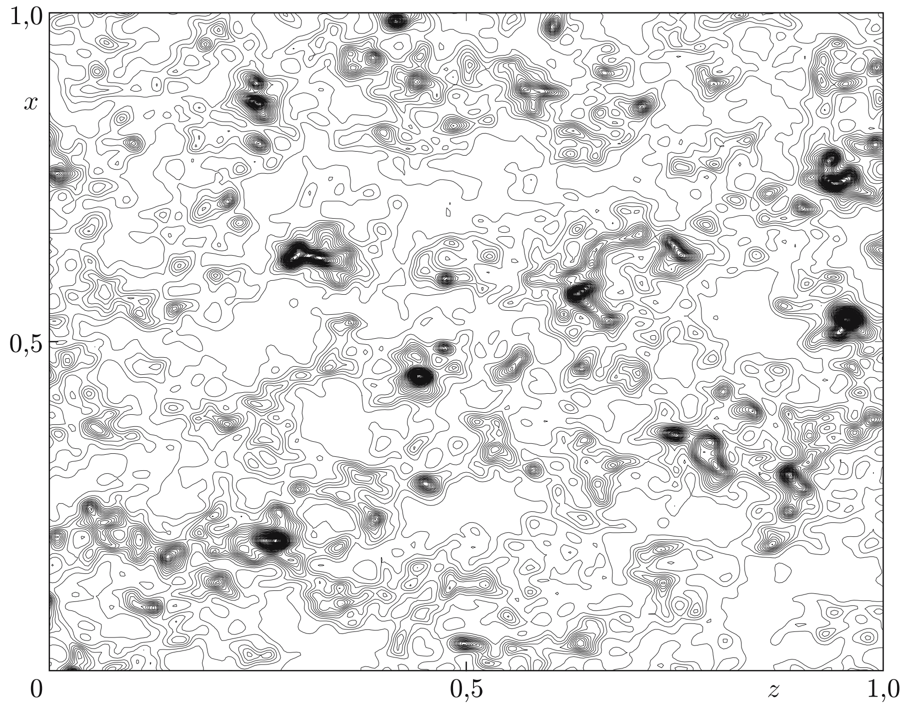

Fig. 2. Permeability isolines for a lognormal law ($\alpha = 2$); the number of levels is 50.

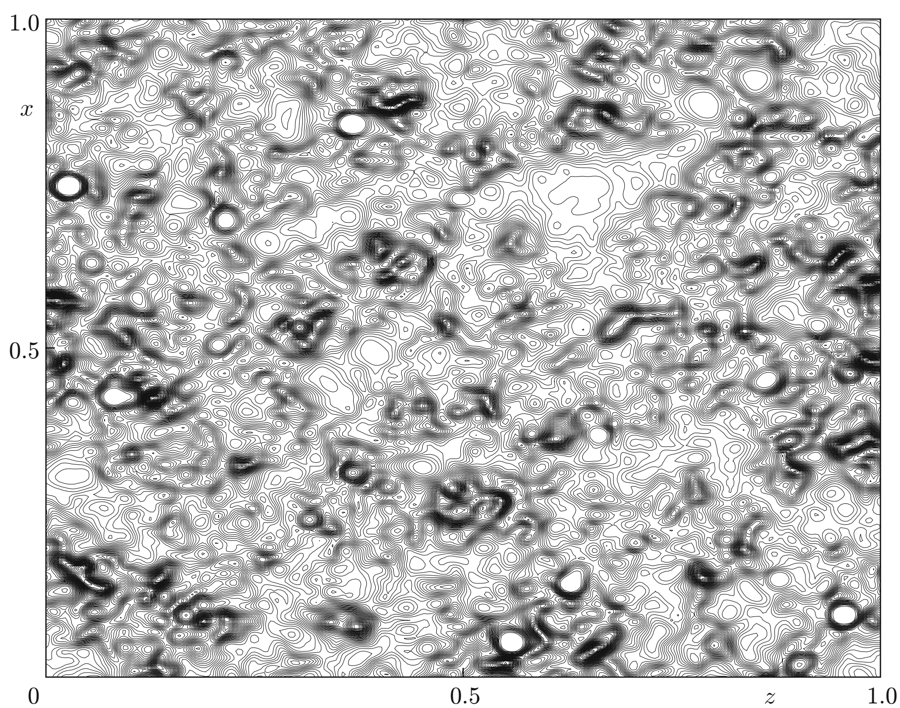

Fig. 3. Permeability isolines for a log-stable law ($\beta = 1$ and $\alpha = 1.6$); the number of levels is 50.

difference problem yields a good approximation of Eq. (1). In formula (4), the independent random variables $\zeta_{\boldsymbol{i}}^l$ are modeled using the algorithm and program given in [17].

For scale-invariant media, the constants Φ_0 and φ_0 can be chosen from experimental data for porous media. According to [15], for some media the exponent in formula (6) is in the range of 0.25–0.3. Figures 2 and 3 give isolines of the scale-invariant permeability in the middle cross section $z = 1/2$ for $\alpha = 2$ (which corresponds to a lognormal permeability model) and for $\alpha = 1.6$ and $\beta = 1$ (which corresponds to a log-stable model). The parameters are $\Phi_0 = 0.3$ and $\varphi_0 = 0$. The difference between the two models is obvious. In Fig. 2, the light (large values) and dark spots (small values) are almost equal in number, whereas in Fig. 3, large values of the permeability function are rare. Because of this property of the stable distributions, statistical processing of results requires a far larger body of data than for normal distributions. Therefore, the calculations are performed for only three layers. For the lognormal permeability model in [4], four layers were used.

According to the procedure of deriving the subgrid formula, in order to verify this formula, it is necessary to solve the full problem numerically and to perform probabilistic averaging over small-scale fluctuations. As a result, one obtains a subgrid term, which can be compared with the theoretical expression. The probabilistic averaging requires a repeated solution of the full problem with a specified large-scale permeability component and a random subgrid component with the subsequent averaging over it. In the present study, a more economical version of the verification was performed. It is based on the power-law dependence of the total flow on the ratio of the maximum and minimum scales in the supergrid region in permeability calculations using formula (4) if the contribution of the subgrid region is ignored. To show this, we perform the following transformations of the derived formulas. The effective permeability (17) should lead to the true velocity in the region (L, l). In particular, the total fluid flow through the sample should coincide with the true value irrespective of the truncation scale of l. For the average fluid flow through the volume in a direction parallel to the coordinate planes for a unit cube, we have

$$\Big\langle \varepsilon_{0l} \exp\Big[-\Big(\ln 2 \sum_{\hat l}^{\hat L} \varphi(\boldsymbol{x}, \tau_l)\delta\tau\Big)\Big] \nabla_i p \Big\rangle = Q_i.$$

Taking into account Eq. (17), we obtain

$$\varepsilon_{0l}\Big\langle \exp\Big[-\Big(\ln 2 \sum_{\hat l}^{\hat L} \varphi(\boldsymbol{x}, \tau_l)\delta\tau\Big)\Big] \nabla_i p \Big\rangle = \varepsilon_{00}\Big(\frac{l}{L}\Big)^{\delta\tau \ln 2 \sum\limits_{\hat l_1 = \hat l}^{\hat L} \Big(\Phi_0(\tau_{\hat l_1})^{\frac{2(1-2^{\alpha-1})+D}{2D\cos(\pi\alpha/2)}} + \varphi_0(\tau_{\hat l_1})\Big)}$$

$$\times \Big\langle \exp\Big[-\Big(\ln 2 \sum_{\hat l_1 = \hat l}^{\hat L} \varphi(\boldsymbol{x}, \tau_{\hat l_1})\delta\tau\Big)\Big] \nabla_i p \Big\rangle = Q_i.$$

Therefore,

$$\Big\langle \exp\Big[-\Big(\ln 2 \sum_{\hat l}^{\hat L} \varphi(\boldsymbol{x}, \tau_l)\delta\tau\Big)\Big] \nabla_i p \Big\rangle = \frac{Q_i}{\varepsilon_{00}}\Big(\frac{l}{L}\Big)^{-\delta\tau \ln 2 \sum\limits_{\hat l_1 = \hat l}^{\hat L} \Big(\Phi_0(\tau_{\hat l_1})^{\frac{2(1-2^{\alpha-1})+D}{2D\cos(\pi\alpha/2)}} + \varphi_0(\tau_{\hat l_1})\Big)}.$$

According to the Darcy law written on the scale $L_0 \gg L$, the total flow is defined by the relation

$$\sum_i^3 Q_i = \varepsilon_{00}\,\frac{p_2 - p_1}{y_2 - y_1}.$$

In the present calculations, it is assumed that the scale L_0 is large enough compared to L and the probabilistic average can be replaced by the average over space (the ergodic hypothesis). Thus, numerical verification needs to be done for the formula

$$\sum_i^3 \Big\langle \exp\Big[-\Big(\ln 2 \sum_{\hat l}^{\hat L} \varphi(\boldsymbol{x}_j, \tau_l)\delta\tau\Big)\Big] \nabla_i p \Big\rangle = \frac{p_2 - p_1}{y_2 - y_1}\Big(\frac{l}{L}\Big)^{-\delta\tau \ln 2 \sum\limits_{\hat l_1 = \hat l}^{\hat L} \Big(\Phi_0(\tau_{\hat l_1})^{\frac{2(1-2^{\alpha-1})+D}{2D\cos(\pi\alpha/2)}} + \varphi_0(\tau_{\hat l_1})\Big)},$$

where the average is understood as the average over space.

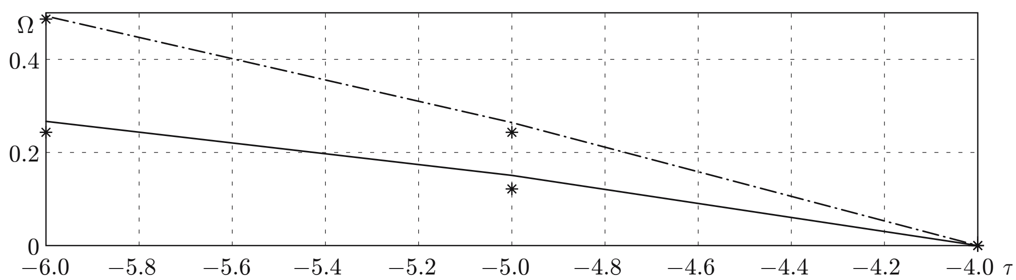

Fig. 4. Curve of $\Omega(\tau)$ for the scale-invariant permeability model: the points refer to numerical modeling, the solid curve refers to the theoretical values obtained using formula (19) for $\Phi_0 = 0.3$, and the dash-and-dotted curve to theoretical values for $\Phi_0 = 0.6$.

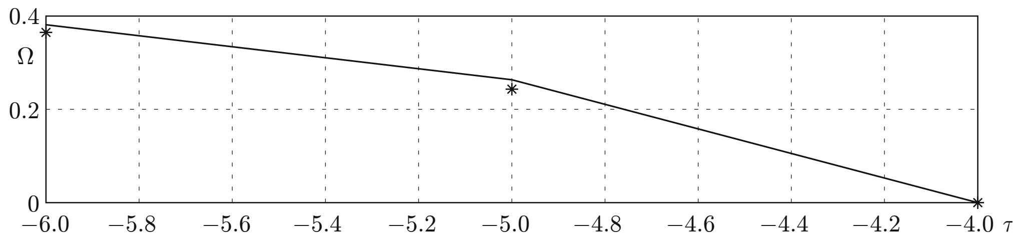

Fig. 5. Curve of $\Omega(\tau)$ for the permeability model which is not scale-invariant: the points refer to numerical modeling, and the solid curves refer to the values obtained using formula (19).

Next, the grid analog of the nondimensional Eq. (1) is solved numerically. The numerical solution is performed using a version of the iteration method [18]. Then, we obtain the left side of formula (19) using the numerical solution of Eq. (1) where the oscillations of the smallest scale are $\varepsilon_{-4}, \ldots, \varepsilon_{-6}$. The maximum scale is $\tau_l = -4$. For the scale-invariant model, the plot has the form of a straight line (because a power-law dependence is obtained) with a slope to the abscissa $\chi = \Phi_0(2(1 - 2^{\alpha-1}) + D)/(2D\cos(\pi\alpha/2)) + \varphi_0$. The results of the numerical verification for $\Phi_0 = 0.3$, $\Phi_0 = 0.6$, and $\varphi_0 = 0$ are given in Fig. 4. The parameters have the values $p_1 - p_2 = 1$ and $y_2 - y_1 = 1$, where $\Omega_{\tau_{\hat{l}}} = \sum_i^3 \log_2 \Big\langle \exp\Big[-\sum_{\hat{l}}^{\hat{L}} \varphi(\boldsymbol{x}_j, \tau_{\hat{l}})\delta\tau\Big] \dfrac{\nabla_i p}{P_2 - P_1} \Big\rangle$ is the logarithm of the flow; $\alpha = 1.6$; $\beta = 1$; and $\tau_{\hat{l}} = -6, -5, -4$. If the parameters Φ_0 and φ_0 depend on the scale, there is no scale invariance and the dependence of the logarithm of the flow on the logarithm of l should be a broken line, each link of which has a slope to the abscissa specified by the values of the parameters $\Phi_0(\tau_l)$ and $\varphi_0(\tau_l)$. The calculation results for the parameter values $\Phi_0(-4) = 0.3$, $\Phi_0(-5) = 0.6$, $\Phi_0(-6) = 0.3$, and $\varphi_0(i) = 0$ are given in Fig. 5. We note that there is good agreement between the results obtained by the subgrid modeling and numerical modeling.

Discussion. In the present study, we obtained formulas for the calculation of the effective permeability (17) that take into account the contribution of the small-scale component to the calculation of the average characteristics, such as, for example, the average fluid flow through an inhomogeneous porous medium. A simple method for calculating the effective coefficients is the use of perturbation theory, in which the solution is represented as a series expansion in the powers of a certain fluctuation amplitude [2]. This approach, as a rule, is confined to only the lowest approximations of perturbation theory. The renormalization group method is applied to improve perturbation theory. As was shown in the present paper, the Wilson RG ideas provide good results not only for lognormal permeability models [4] but also for permeability models with log-stable laws. The effective coefficients were obtained for stable distributions of parameters for $1 < \alpha \leq 2$ and $\beta = 1$. This is explained by the fact that in this study we used correlation functions. If $-1 < \beta < 1$, the second moments of the permeability function do not exist and the problem can be solved only using probability distribution functions. For stable distributions, explicit analytical expressions for the probability densities (with rare exceptions) are absent. The version considered in the present paper has been supported experimentally [1]. In the case of scale similarity in the medium, the effective transformation formula (17) contains constants on the right side. In the absence of scale similarity, the constants are replaced by functions of the scale. If these functions of the scale are known (for example, from empirical data), the derived formulas are also useful in the general case of log-stable inhomogeneous media. Direct numerical verification using spatial averaging gave good agreement between the theoretical formulas and results of numerical experiments.

Appendix. Let us calculate the statistical average permeability fields. The characteristic function of a random variable in form (B) [13] is written as

$$\ln \chi(t) = \sigma_B(it\mu_B - |t|^\alpha \omega_B(t,\alpha,\beta_B)),$$

where

$$\omega_B(t,\alpha,\beta_B) = \begin{cases} \exp(-i(\pi/2)\beta_B(\alpha-2)\,\mathrm{sign}\,(t)), & \alpha > 1, \\ -\beta_B(2/\pi)\ln|t|, & \alpha = 1. \end{cases}$$

Let $\alpha \neq 1$ and $g(t,\alpha,\beta_B)$ be the probability density of the random variable ζ. The statistical average of the random variable $\mathrm{e}^{-b\zeta}$ is given by

$$\langle \exp(-b\zeta)\rangle = \int\limits_{-\infty}^{\infty} \exp(-bt) g(t,\alpha,\beta_B)\,dt = \int\limits_{0}^{\infty} \exp(-bt) g(t,\alpha,\beta_B)\,dt + \int\limits_{0}^{\infty} \exp(bt) g(t,\alpha,-\beta_B)\,dt. \tag{A1}$$

The first integral converges for $\beta_B = 1$ and $\alpha \geq 1$ since the integrand has the coefficient $\exp(-bt)$ and $g(t,\alpha,\beta_B)$ has the asymptotic expansion

$$g(t,\alpha,1) \sim \frac{\alpha}{\pi t}\sum_{k=1}^{\infty}\frac{\Gamma(\alpha k)}{\Gamma(k)}\,t^{-k\alpha}\sin[\pi k(2-\alpha)], \qquad x \to \infty, \quad \alpha > 1,$$

$$g(t,\alpha,1) \sim \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k!}\,P_k(\ln x)x^{-k-1},$$

where P_k are polynomials of order k. The second integral converges for $\alpha \geq 1$ since $g(t,\alpha,-1)$ has the asymptotic expansion

$$g(t,\alpha,-1) \sim \frac{(t/\alpha)^{(2-\alpha)/(2(\alpha-1))}}{\sqrt{2\pi\alpha|1-\alpha|}}\,\exp\Big\{-|1-\alpha|\Big(\frac{t}{\alpha}\Big)^{\alpha/(\alpha-1)}\Big\}, \qquad x \to \infty.$$

For $0 < \alpha < 1$, all values of the random variable ζ are positive, and, therefore, the integral (A1) converges. Since the convergence of the integral (A1) is proved, we can introduce the factor $\exp[-(x/d)^2]$ $(d \to \infty)$ into the integrand and employ the saddle point method [19] (the saddle point $t_0 = ib$). As a result, we have

$$\langle \exp(-b\zeta)\rangle = \exp[-\sigma_B\mu_B b + \sigma_B b^\alpha]. \tag{A2}$$

Using formula (A2) and taking into account that $\mu = 0$, $\sigma = 1$, $\beta = 1$, and $1 < \alpha < 2$ for $\zeta_{\boldsymbol{k}}$ [the parameters in form B are linked to the parameters in form A by the relation $\beta = \beta_B = 1$, $\mu = \mu_B/\cos(\pi\alpha/2)$, $\sigma = \sigma_B\cos(\pi\alpha/2)$], for $\varepsilon(\boldsymbol{x})$, we obtain

$$\begin{aligned}
\langle\varepsilon(\boldsymbol{x})\rangle &= \varepsilon_0\Big\langle \exp\Big[-\delta\tau\ln 2\Big(\sum_{\hat{l}=\hat{l}_0}^{\hat{L}}\sum_{\boldsymbol{k}}\Big(\frac{\Phi_0(\tau_{\hat{l}})}{2(\delta\tau\ln 2)^{\alpha-1}}\Big)^{1/\alpha} a^l_{\boldsymbol{k}}\zeta_{\boldsymbol{k}} + \varphi_0(\tau_{\hat{l}})\Big)\Big]\Big\rangle \\
&= \prod_{l=\hat{l}_0}^{\hat{L}}\Big\langle \exp\Big[-\delta\tau\ln 2\Big(\sum_{\boldsymbol{k}}\Big(\frac{\Phi_0(\tau_{\hat{l}})}{2(\delta\tau\ln 2)^{\alpha-1}}\Big)^{1/\alpha} a^l_{\boldsymbol{k}}\zeta_{\boldsymbol{k}} + \varphi_0(\tau_{\hat{l}})\Big)\Big]\Big\rangle \\
&= \prod_{l=\hat{l}_0}^{\hat{L}}\exp\Big[\frac{\Phi_0(\tau_{\hat{l}})\delta\tau\ln 2}{2}\Big[\cos\Big(\frac{\pi\alpha}{2}\Big)\Big]^{-1}\sum_{\boldsymbol{k}}(a^l_{\boldsymbol{k}})^\alpha - \varphi_0(\tau_{\hat{l}})\delta\tau\ln 2\Big] \\
&= \varepsilon_0\exp\Big[-\frac{\delta\tau\ln 2}{2}\Big[\cos\Big(\frac{\pi\alpha}{2}\Big)\Big]^{-1}\sum_{\hat{l}=\hat{l}_0}^{\hat{L}}\Phi_0(\tau_{\hat{l}}) - \varphi_0(\tau_{\hat{l}})\delta\tau\ln 2\Big].
\end{aligned} \tag{A3}$$

If the medium is scale-invariant, $\varphi_0(l)$ and $\Phi_0(l)$ do not depend on l and the average $\varepsilon(\boldsymbol{x})$ is given by the formula

$$\langle\varepsilon(\boldsymbol{x})\rangle = \varepsilon_0 \exp\left[-(\Phi_0/(2\cos(\pi\alpha/2)) + \varphi_0)(\ln L - \ln l_0)\right] = \varepsilon_0 (L/l_0)^{-(\Phi_0/(2\cos(\pi\alpha/2))+\varphi_0)}.$$

Let us calculate the second single-point moment for $\varepsilon(\boldsymbol{x})$ using formula (A2) (stable-distribution parameters in form A):

$$\langle\varepsilon^2(\boldsymbol{x})\rangle = \varepsilon_0^2 \Big\langle \exp\Big[-\sum_{\hat{l}_0}^{\hat{L}} 2\varphi(\boldsymbol{x}, \tau_l)\delta\tau\ln 2\Big]\Big\rangle = \varepsilon_0^2 \exp\Big[-2\delta\tau\ln 2\Big(2^{\alpha-2}\Big[\cos\Big(\frac{\pi\alpha}{2}\Big)\Big]^{-1}\sum_{\hat{l}=\hat{l}_0}^{\hat{L}}\Phi_0(\tau_{\hat{l}}) + \varphi_0(\tau_{\hat{l}})\Big)\Big].$$

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